Dynamics of pinned interfaces with inertia

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We introduce velocity dependent pinning into two models which are known to be in the universality class of the directed percolation depinning (DPD) model. The effective internal force acting on any point of the interface is enchanced by a factor f at that point of the interface which has last moved. This causes the effective roughness exponent to cross over continuously from the DPD value of 0.63, to unity in the nondissipative limit of $f\rightarrow\infty$, while the growth exponent tends to 3/4. DPD scaling is recovered for length scales above a persistence length which grows with the enchancement factor as f^{ψ}, with a new exponent $\psi\simeq 1.3$.
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The dynamics of interfaces in far-from-equilibrium systems has attracted a great deal of attention lately. Rough interfaces in systems with quenched disorder [1,2] arise in phenomena as diverse as the imbibition of liquids by porous media [3] and the evolution of species [4]. One feature of obvious interest in the critical dynamics of driven interfaces [5] is the velocity of the interface, how it behaves with the driving force near the threshold, and how and whether it depends on the roughness (or conversely the stiffness) of the interface and vice versa. On the other hand, for invasion percolation and other discrete models which can be derived from it, such as the Sneppen model [6], there is no tunable parameter corresponding to a driving force. The front, subject to various constraints (as, for example, the constraint that the slopes not exceed unity in absolute value, in the Sneppen model) advances with constant average velocity.

In this paper we introduce an inertial component into the Sneppen model [6] and a variant of the directed percolation depinning model [7–9], by giving that interfacial point which has last moved an enhanced probability to keep on moving. This can also be regarded as modeling a velocity dependence in the dissipation or the pinning strength in the system, as for example in shear thinning observed in complex fluids [10]. We investigate how the mean velocity and the roughness exponent in these systems, both known to be in the same universality class in 1+1 dimensions [11] depend upon this enchancement factor.

The nonlinear dissipative equation of motion in one dimension,

$$m\frac{\partial^2 h(x,t)}{\partial t^2}=-\gamma\frac{\partial h(x,t)}{\partial t}+\tilde{\nu}\nabla^2 h+\frac{\tilde{\lambda}}{2}\left(\frac{\partial h}{\partial x}\right)^2+\tilde{\eta}(x,h)+\tilde{F}, \tag{1}$$

where γ, $\tilde{\nu}$, and $\tilde{\lambda}$ are constants and $\tilde{\eta}$ a δ-correlated noise term, can be examined under simplifying assumptions in order to understand the limiting behaviors. In the absence of noise, and in the limit of $\nu=0$, $\tilde{\lambda}=0$, this equation can easily be solved to show that after a transient regime with exponential relaxation, with the relaxation time $\tau=m/\gamma$, the average position of the front becomes a linear function of time, i.e., the velocity saturates to a constant value $v_f=\tilde{F}/\gamma$. In other words, the effect of the inertial term is only transient, and the final velocity is independent of m. Dividing through by γ and assuming from the start that the coefficient of the acceleration is negligibly small, one ends up with the Kardar, Parisi, Zhang (KPZ) equation with quenched noise [12,13], which is commonly believed [2] to correspond to the continuum version (or long wavelength limit) of driven interfaces in quenched random media.

$$\frac{\partial h(x,t)}{\partial t}=\nu\nabla^2 h+\frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2+\eta(x,h)+F. \tag{2}$$

In the presence of the nonlinearity ($\lambda\neq 0$), with surface tension $\nu\neq 0$, numerically integrating Eq. (2), one finds that the velocity of the center of mass of the interface, $v=\partial\overline{h}/\partial t$ saturates to a constant value, which is dependent only on F, the ''constant drive'' [5,14] but independent of ν, i.e., the stiffness of the interface.

Now we turn to discrete, self-organized models, the Sneppen model [6] and the directed percolation depinning (DPD) model [7–9]. We remind the reader that the Sneppen model [6] is defined via the following growth rules (i) conduct a nonlocal search for the largest value of the quenched random variable $\eta(x,h)$ and move the interface by one unit at the value of x which maximizes η. (ii) Then check the slopes on either side of point that has just moved; move the neighboring points as well, until all slopes are $\leqslant 1$ in absolute value. The motion of many such spatially contiguous sites is termed an ''avalanche.'' This is a model of the ''constant current'' type, where ''time'' is advanced by one step every time any point on the interface is moved by one lattice spacing, so that the interface velocity is constant. No satisfactory continuum limit exists for such models, although it has been argued [2] that it is nevertheless described by an equation of type (2) where $F=0$ and $\partial h/\partial t\neq 0$ only at that x where

$$F_{\rm int}=\nu\nabla^2 h+\frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2+\eta(x,h) \tag{3}$$

has a maximum.

We have also studied the directed percolation depinning model, where no constraint is introduced on the slopes. The growth site chosen by the maximization process can neighbor the interface either in the forward direction, or can be to its right or left. In the latter two cases, the interface is updated by filling that site and all the sites below it.

We now introduce an ''inertial effect'' into both dynamics, by allowing the motion not to be damped to zero at one time step, but giving that point which has last triggered an event an advantage, so that it can keep on moving in subsequent time steps. We do this by multiplying the quenched random variables neighboring the growth site which was last selected by a factor $f>1$. It can be seen that in the continuum language, this is equivalent to introducing a velocity-dependent coefficient before the noise term in Eq. (3), viz.,

$$F_{\rm int}=\nu\nabla^2 h+\frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 +\left[1+(f-1)\Theta\left(\frac{\partial h}{\partial t}\right)\right]\eta(x,h), \tag{4}$$

where Θ is the step function, with $\Theta(0)\leqslant 0$ and $\Theta(s)=1$ for $s>0$. We have also investigated the effect of choosing $f<1$, namely discouraging points which have last spearheaded an event. Both these regimes are of physical interest, since, for example both shear thinning and shear thickening are observed in nonlinear flow behavior [10]. The parameter f can be viewed as interpolating between the perfectly dissipative ($f\to 0$) and nondissipative ($f\to\infty$) limits of the dynamics. In the nondissipative limit, once an avalanche is started it persists forever, or, in other words, the interface is no longer pinned.

A natural reinterpretation of the dynamics is to keep the time fixed until an event, triggered by the quenched random variable η at any one point, terminates. This modification in the definition of time does not lead to new dynamics in the long time limit; although the average velocity exhibits fluctuations on small time scales, one still has $\overline{h}=vt+\text{const}\,t$ over large time scales, as can be seen in Fig. 1, and the ''constant current'' picture still applies on the average. The fluctuations about the mean velocity grow with f (see Fig. 1(b) and (c). The mean velocity of the front, v, grows linearly with f for small f, eventually saturating to 1 for $f\to\infty$ (see Fig. 2) (Since there is no driving force on the system, v remains a constant in the nondissipative limit.)

We now consider the effect that ''inertia'' has on the roughness of the interface, in the case of the Sneppen model. Recall that on a discrete one-dimensional lattice of length L, the width of the interface over an interval of $\ell<L$ scales as

$$W=\left\{\frac{1}{\ell}\sum_i^{\ell}\left[h(i)-\overline{h}\right]\right\}^{1/2}\sim\ell^{\chi}g\left(\frac{t}{\ell^z}\right), \tag{5}$$

where χ is the roughness exponent, z is the dynamical exponent, the crossover function $g(x)\sim$const for $x\gg 1$ and $g(x)\sim x^{\beta}$ for $x<1$, with the scaling relation for the growth exponent, $\beta=\chi/z$. For the default value of $f=1$, the values of these exponents are known both from extensive simulations and their relation to directed percolation exponents in $1+1$ dimensions, and $\chi=0.633$, $\beta=0.63$ for relatively short times, crossing over to 0.9 ± 0.1 for longer times [2,6,7,9].

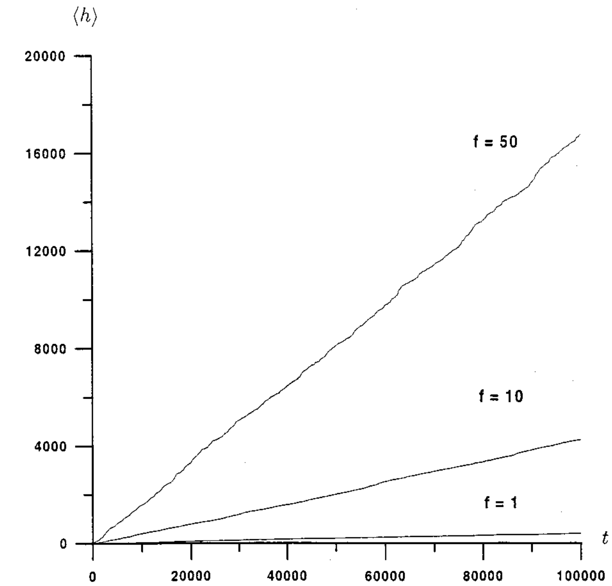

FIG. 1. In the modified Sneppen model, the center of mass of the interface moves with a constant velocity on the average. The fluctuations are related to the avalanche size distribution. (a) $f=1$, (b) $f=10$, (c) $f=50$.

Our simulations on chains of length $L=16$ to 1024, with periodic boundary conditions show that the value of χ as defined in Eq. (3) with $\ell=L$ depends continuously on f, saturating to the trivial value of 1 in the $\lim f\to\infty$. Our results are shown in Fig. 3, where the log-log plot shows the dependence of $\chi_{\rm eff}$ on f. An average of over 20 snapshots for

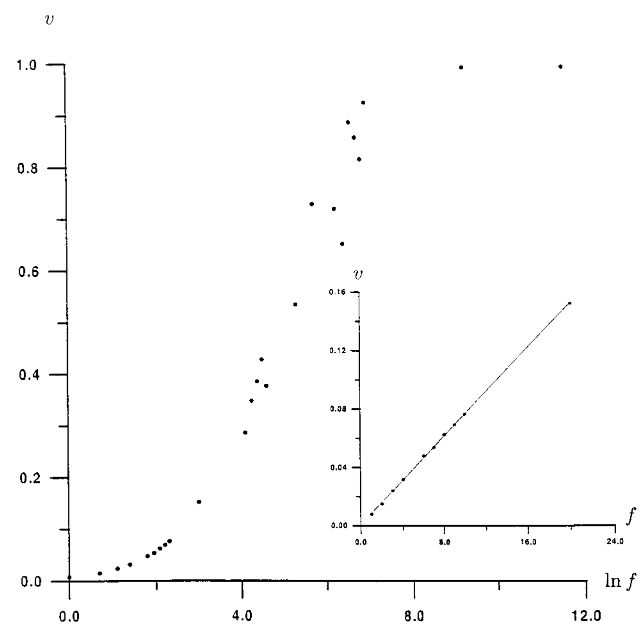

FIG. 2. The average interface velocity as a function of the inertial parameter f. Averages have been performed over 50 runs, on systems of size $L=512$ and 1024. The fluctuations, which are much smaller for small and for extremely large values of f, are seen to persist for intermediate values of f. The inset shows the linear behavior of v with small values of f.

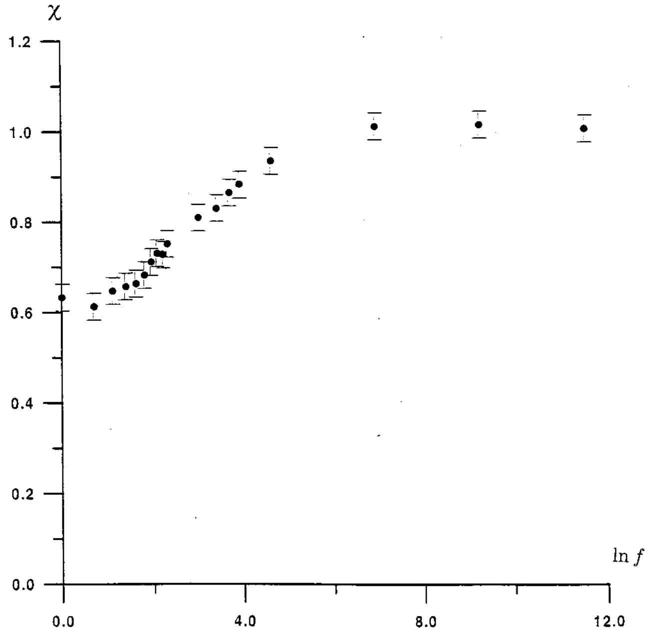

FIG. 3. The effective roughness exponent as a function of *f*.

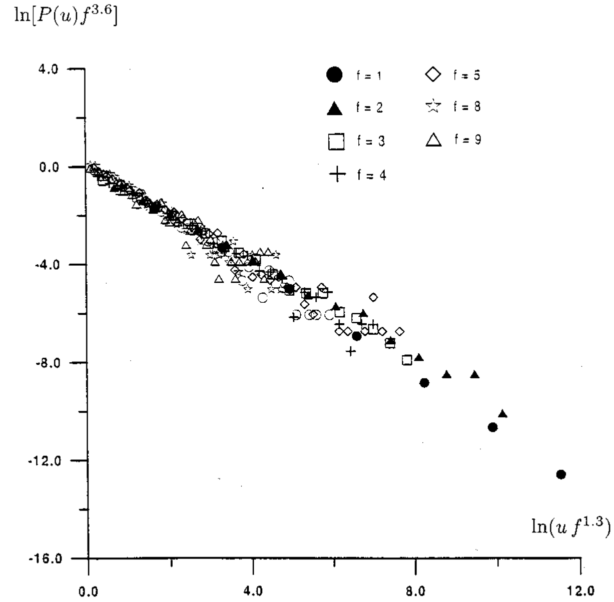

FIG. 4. The distribution function for the persistence length. Shown is the double logarithmic plot of $P(u)/f^{-3.6}$ vs $uf^{1.3}$, for different values of *f*.

each L, separated by times longer than decorrelation times, have been performed.

The limit of $f \rightarrow \infty$ is easy to understand, since, in this limit, there is only one finger which organizes the interface into the shape of an isoceles triangle, with the height simply proportional to the length of the base. For $f<1$, the dynamics is essentially unmodified, since only those rare events are suppressed, which correspond to choosing twice in succession the same point along the interface. One sees from Fig. 3 that χ_{eff} barely deviates from its Sneppen value for $f \rightarrow 0$.

A closer examination reveals that for each *f*, a finite length scale is introduced into the problem via the average longitudinal persistence length, ξ_u, defined as the average number of successive times a given tip is advanced. The persistence length u obeys an exponential distribution as shown in Fig. 4,

$$P(u) \sim A(f)\exp(-u/\xi_u), \tag{6}$$

with $\xi_u \sim f^{\psi}$. We find that $\psi = 1.3 \pm 0.1$, and $A(f) \sim f^{-3.6}$ for sufficiently large $L \gg f^{\psi}$. For finite systems and large *f* such that $\xi_u \sim f^{\psi} \gg L/\sqrt{2}$, the persistence length is effectively infinite; there is simply one peak which grows uninterruptedly.

The crossover behavior is summarized by the scaling relation

$$W \sim f^{\psi} g\left(\frac{\ell}{\xi_u}, \frac{L}{\xi_u}\right), \tag{7}$$

where

$$g(x,y) \sim \begin{cases} y; & y \ll 1 \\ x^{\chi}; & x \gg 1, \quad y \gg 1, \\ x; & x \ll 1, \quad y \gg 1 \end{cases} \tag{8}$$

and χ takes the Sneppen (DPD) value of 0.63. In Fig. 5, we have plotted w/f^{ψ} vs ℓ/f^{ψ} for fixed L and different *f* on a double logarithmic plot. The results are for late times, averaged over 50 independent runs for $L = 1024$ up to 6144, and $2 \leqslant \ell \leqslant L$. One sees that due to the coarse graining of the interface up to length scales comparable with the persistence length, one has $W \sim \ell$ for $\ell < \xi_u$, whereas at scales larger than the persistence length and sufficiently smaller than L, the data collapse for different *f* has a common slope of 0.63.

The exponent β for the initial growth of the interface roughness with time can be obtained exactly in the limits of $f \rightarrow 0$ and $f \rightarrow \infty$. When $f \ll 1$, for very early times, $\overline{h} \sim \overline{h^2} \sim \Sigma_i^t 1/L \sim t/L$, yielding $W \sim t^{1/2}$, or $\beta = 1/2$. On the other

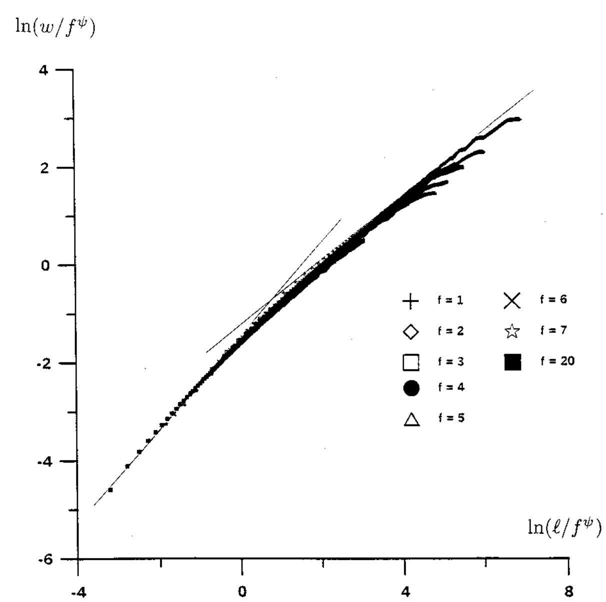

FIG. 5. The scaling function for the width, showing data collapse for different values of *f*. One has $w/f^{\psi} \sim \ell$ for sufficiently large *f*, while $w/f^{\psi} \sim \ell^{\chi}/f^{\chi\psi}$ for $\ell > f^{\psi}$, with $\psi \simeq 1.3$ and $\chi \simeq 0.63$.

hand, for $f \gg 1$ we have $W^2 = (2/L)\Sigma_i^u i^2 - [(2/L)\Sigma_i^u i]^2$, where u is equal to the height of the finger in the shape of an isoceles triangle. Since $t \simeq u^2$, we have, for $u/L \ll 1$, $W \sim t^{3/4}$, or $\beta = 3/4$.

Finally, it is interesting to remark that for $f \gg 1$, the growth of the interface is decorated with oscillations which arise trivially from the constraint on the slopes; each advance of the tip of a finger is accompanied by avalanches of duration u that run down both sides of the triangle. Note that u can grow by unity after $T = 2u$ time steps. The ''period'' of the nth oscillation is $T_n = 2n$, for $n \leqslant L/2$. For $T_n \leqslant t \leqslant T_{n+1}$,

$$W(u,t) - W_0 = \frac{2}{u}\left(1 + \frac{2}{u}\right)\left[\left(\frac{u}{4}\right)^2 - \left(t - \frac{u}{4}\right)^2\right], \qquad (9)$$

where $W_0 = u^2/48$. For large times $t > L^2/4$, there is only one finger with u fixed at $L/2$ and the oscillations in Eq. (9) become strictly periodic.

In conclusion, we have presented an extension of the Sneppen model by introducing a velocity dependent pinning, or equivalently an inertia effect. This leads to a characteristic length scale in the problem, a persistence length which depends upon the inertial parameter through a scaling law, $\xi_u \sim f^{\psi}$, with $\psi \simeq 1.3$, and causes a crossover to a different scaling regime in the limit of $f \to \infty$. Previously, Roux and Hansen studied a model [15] where they allowed the growth probability to depend on the local curvature, and found a continuous dependence of the effective roughness exponents on this weighting parameter. Since increasing f gives an advantage to growth at the tips, this persistence effect is indeed similar to a ''curvature driven'' [15,16] growth, although $\nu > 0$ in Eq. (4). However, we are now able to understand the parameter dependence of the exponents in terms of a crossover phenomenon. The persistence results in coarse graining of the surface up to scales $\ell < f^{\psi}$, whereas Sneppen (DPD) scaling behavior is restored at larger scales.

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